Tutorial on Scaling Analysis of Navier-Stokes Equations: Linear and Non-linear Dynamics of Fluid-Structure-Interaction

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1. Introduction

In an earlier paper, a method called scaling analysis has been used to extract useful and fundamental information from the unsteady transonic small disturbance potential flow model [1]. Here the same methodology is applied to the Navier-Stokes equations. Scaling analysis has also been used for a variety of nonlinear dynamics models in aeroelasticity including the modeling of structural nonlinearities [2, 3, 4] as well as in the field of thermodynamics and heat transfer [5]. In scaling analysis one does not seek to find a solution to the mathematical model in the conventional sense, but rather to make order of magnitude estimates for the likely outcome of computational solutions. The estimates are made using analytical techniques which do not require any substantial numerical effort. Their purpose and advantage is to provide a benchmark for the expected results of computational studies and also to estimate the relative importance of various effects that could (or could not) be included in theoretical models.

2. The Navier-Stokes Equations

Consider the vector momentum equation for a fluid described by the Navier-Stoke equations as follows. For our purposes here, it is sufficient to consider a two-dimensional flow.

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla) \boldsymbol{v} = -\frac{1}{\rho} \nabla p + \frac{\mu}{\rho} \nabla^2 \boldsymbol{v}$$
(1)

More specifically consider the streamwise scalar component of this equation where x and y are the usual cartesian coordinates.

$$\rho\left[\underbrace{\frac{\partial u}{\partial t}}_{(a)} + \underbrace{u\frac{\partial u}{\partial x}}_{(b)} + \underbrace{v\frac{\partial u}{\partial y}}_{(c)}\right] = \underbrace{-\frac{\partial p}{\partial x}}_{(d)} + \mu\left[\underbrace{\frac{\partial^2 u}{\partial x^2}}_{(e)} + \underbrace{\frac{\partial^2 u}{\partial y^2}}_{(f)}\right]$$
(2)

The various terms in Eq. 2 are numbered (a) to (f).

Scaling analysis has also been applied to the transverse scalar component of Eq. 1. However the outcome is simply to reproduce the known results of boundary layer theory. See the appendix. For our purposes it is assumed that the density can be estimated to be its free stream value or if the reader prefers the flow is assumed to be one of constant density. As will become clear, this estimate for the density is sufficient for the present scaling analysis even in compressible flow. Thus, the density ρ and the streamwise scalar component of the flow velocity u, may be expressed as follows.

$$u = U_{\infty} + \hat{u}$$
 and $\rho = \rho_{\infty}$ (3)

where U_{∞} is the free stream velocity and \hat{u} is a (small or large) perturbation.

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3. Scaling Analysis

Now to begin scaling analysis per se, the spatial and time derivatives are assumed to be of the following order where D is a characteristic length in the streamwise direction, λ is a characteristic length in the transverse direction and ω is a characteristic frequency of unsteady oscillations in the flow. D may be typically the chord of an airfoil or wing and ω may be prescribed or determined as an outcome of the scaling analysis depending on the physical phenomena of interest. λ is usually found as an outcome of the scaling analysis.

$$\frac{\partial}{\partial x} \sim \frac{1}{D}, \qquad \frac{\partial}{\partial x} \sim \omega, \qquad \frac{\partial}{\partial y} \sim \frac{1}{\lambda}$$
 (4)

Scaling analysis is performed by asking the question: how do the several terms in the mathematical model (here the Navier-Stokes equations and more particularly Eq. 2) balance? By balancing it is meant that the two terms are of the same order of magnitude and one may not be neglected relative to the other. If two terms do not balance, then one may be neglected compared to the other.

3.1 Balancing terms (a) and (b)

So, for example, if one requires that terms (a) and (b) balance in Eq. 2, then

$$\omega \hat{u} \sim U_{\infty} \frac{\hat{u}}{D} \longrightarrow \frac{\omega D}{U_{\infty}} \sim 1$$
 (5)

Now how does one interpret Eq. 5? First of all Eq. 5 agrees with a wide range of experimental evidence including the classical example of the oscillating flow in the wake behind a blunt body (Von Karman vortex street) as well as more recent results for transonic buffeting flows. Conversely if one accepts Eq. 5 as a result of experiment it tells us that terms (a) and (b) balance or are of equal importance for such flows. Of course Eq. 5 also allows one to determine ω given D or vice versa and this well known result has been useful to engineers and scientists for many years.

3.2 Balancing terms (b) and (c)

From this balancing it is determined that

$$U_{\infty}\frac{\widehat{u}}{D} \sim \widehat{v}\frac{\widehat{u}}{\lambda} \longrightarrow \frac{\widehat{v}}{U_{\infty}} \sim \frac{\lambda}{D}$$
 (6)

But recall λ is not yet known and is yet to be determined.

3.3 Balancing terms (b) and (e)

$$\rho_{\infty}U_{\infty}\frac{\widehat{u}}{D} \sim \mu \frac{\widehat{u}}{D^2} \longrightarrow Re \equiv \frac{\rho_{\infty}U_{\infty}D}{\mu} \sim 1$$
(7)

The interpretation of this result is that if the Reynolds number, Re, is of order one, then terms (b) and (e) balance and both are equally important in the theoretical model. However if Re is much greater than one, then term (b) is much greater than (e) and the latter may be neglected. Of course it is the latter case that is most interest in aerospace applications, so consider this case further.

3.4 Balancing terms (b) and (f)

From this balancing one determines that

$$p_{\infty}U_{\infty}\frac{\widehat{u}}{D} \sim \mu \frac{\widehat{u}}{\lambda^2}$$
 (8)

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and "solving" Eq. 8 one determines that

$$\frac{\lambda}{D} \sim \frac{1}{\sqrt{Re}} \tag{9}$$

Note also that Eq. 9 gives the desired estimate for λ . For $Re \gg 1$, $\lambda/D \ll 1$ and Eq. 9 determines λ for $Re \gg 1$!

Again it should be emphasized that Eq. 8 and Eq. 9 are order of magnitude estimates rather than conventional equations per se. Note that if Re is much greater than one then λ is much less D.

3.5 Balancing terms (b) and (d)

From this one determines that

$$\rho_{\infty} U_{\infty} \frac{\widehat{u}}{D} \sim \frac{p}{D} \tag{10}$$

But how to estimate \hat{u} ? Returning to Eq. 3 and considering term (b), it is clear that when the flow oscillates in a nonlinear limit cycle oscillation (in the absence of any structural body motion)

$$\hat{\iota} \sim U_{\infty} \tag{11}$$

when the two components of term (b) are comparable, i.e.

$$(U_{\infty} + \hat{u})\frac{\partial \hat{u}}{\partial x} = U_{\infty}\frac{\partial \hat{u}}{\partial x} + \hat{u}\frac{\partial \hat{u}}{\partial x}$$
(12)

are comparable. Using Eq. 12 in Eq. 11, then

$$\widehat{p} \sim \rho_{\infty} U_{\infty}^2 \tag{13}$$

and thus

$$C_L \sim \frac{2\hat{p}D}{\rho_\infty U_\infty^2 D} \sim 1 \tag{14}$$

Note that Eq. 14 is a prediction and that factors of 2 are not retained in an order of magnitude analysis, i.e. scaling analysis. That is, it is expected that a computational solution to the Navier-Stokes equations will give a lift coefficient for the oscillating lift on a bluff body of order one. In fact this is what is found from a computational fluid dynamics solution, see [6]. This is also the order of magnitude of the oscillating lift found in transonic buffeting flows [7].

3.6 Oscillating Bodies in a Flowing Fluid

Another prediction of scaling analysis is an order of magnitude estimate of the amplitude of an oscillating body or pitching that will induce nonlinear dynamic effects when the body is placed in a flowing fluid. The body may have prescribed motion or it may freely interact with the flow such that a self excited nonlinear oscillation (limit cycle oscillation) of the body interacting with the fluid occurs.

Consider an oscillating body with frequency ω and deflection amplitude A. Thus the body (transverse) velocity w, will be of order

$$v \sim \omega A \tag{15}$$

Analogously for a pitching airfoil.

$$w \sim U_{\infty} \alpha$$
 (16)

The fluid boundary condition requires that the fluid transverse velocity v be equal to the body transverse velocity w. Recalling that the Strouhal number is of order one, see Eq. 5, this implies that the direct and convective time derivative terms in the boundary condition are of the same order. Thus

$$\widehat{v} \sim w \tag{17}$$

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3.7 Bluff Body Motion

Now using Eq. 6, 15 and 17, one can predict the order of magnitude of how large the body motion must be to observe nonlinear dynamic effects in the fluid oscillations due to body motion.

$$\omega A \sim U_{\infty} \frac{\lambda}{D} \sim U_{\infty} \frac{1}{\sqrt{Re}} \tag{18}$$

and thus

$$\frac{A}{D} \sim \frac{U_{\infty}}{\omega D \sqrt{Re}} \tag{19}$$

Using Eq. 5

$$\frac{A}{D} \sim \frac{1}{\sqrt{Re}} \sim \frac{1}{10} \quad \text{for} \quad Re \sim 100 \tag{20}$$

which agrees with experiment for the Von Karman vortex street and its interaction with an oscillating body [6]. Eq. 20 actually offers two predictions. It is an order of magnitude estimate of a) the amplitude the body must move to "lock in" the fluid oscillation frequency to the frequency of the body undergoing prescribed motion and b) also the amplitude of the self excited limit cycle oscillation when the body is free to move. Again see [6]. Also see the recent computer simulations of Raveh [7] for lock in and LCO in transonic buffeting flows. These computational results are also in agreement with the current scaling analysis estimates.

3.8 Pitching Airfoil or Wing

Now using Eq. 6, 15 and 16, one can predict the order of magnitude of how large the airfoil motion must be for nonlinear dynamics effects in the fluid oscillations to be observed.

$$\widehat{v} \sim U_{\infty} \frac{1}{\sqrt{Re}} \tag{21}$$

For the example of interest here Eq. 21 may be expressed in terms of a nondimensional acceleration as follows.

$$\frac{\widehat{v}\omega}{g} \sim \frac{U_{\infty}\omega}{g\sqrt{Re}} \tag{22}$$

To compare this result with experiment, we consider the limit cycle oscillations (LCO) of the F-16 aircraft. We choose typical values as follows: free stream velocity of 1000 ft/sec, frequency of 8 Hz and Re of 10,000,000. The nondimensional acceleration order of magnitude estimate of 0.5 from Eq. 22 is in good correspondence with measured values from flight tests [8].

4. Conclusions

Scaling analysis has been applied to the Navier-Stokes equations to obtain order of magnitude estimates for various quantities of physical interest. This extends earlier scaling analysis work on the transonic potential flow equations. Estimates are obtained for the Strouhal number of oscillating flows, the magnitude of oscillating lift on a body in a vortex street or buffeting flow, and the amplitude that a body must oscillate to provide a qualitative change in such flows or the amplitude of oscillation of a body that will result from the limit cycle oscillations that may occur due to fluid nonlinearities. Also well known results such as the relative importance of various terms in the governing Navier-Stokes are reproduced by scaling analysis as well. Finally the present results have been extended formally to compressible flows which support the results of this paper as well as give additional results. These results will be reported separately.

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A Appendix

In this appendix scaling analysis is applied to the transverse component of Eq. 1 and it is seen the result for large Re is the same as that of classical boundary layer theory. The scalar component of Eq. 1 is as follows.

$$\rho\left[\underbrace{\frac{\partial v}{\partial t}}_{(a*)} + \underbrace{u\frac{\partial v}{\partial x}}_{(b*)} + \underbrace{v\frac{\partial v}{\partial y}}_{(c*)}\right] = \underbrace{-\frac{\partial p}{\partial y}}_{(d*)} + \mu\left[\underbrace{\frac{\partial^2 v}{\partial x^2}}_{(e*)} + \underbrace{\frac{\partial^2 v}{\partial y^2}}_{(f*)}\right]$$
(23)

The terms of interest are labeled (a^*) to (f^*)

A1 Balancing terms (a^{*}) and (b^{*})

$$\omega \hat{v} \sim U_{\infty} \frac{\hat{v}}{D} \longrightarrow \frac{\omega D}{U_{\infty}} \sim 1$$
 (24)

which is the same as Eq. 5 of course.

A2 Balancing terms (a^{*}) and (c^{*})

$$\omega \widehat{v} \sim \frac{\widehat{v}}{\lambda} \widehat{v} \quad \text{or} \quad \frac{\omega D}{U_{\infty}} \sim \frac{\widehat{v} D}{U_{\infty} \lambda} \quad \text{or} \quad \frac{\widehat{v}}{U_{\infty}} \sim \frac{\lambda}{D}$$
(25)

which is the same as Eq. 6 of course.

A3 Balancing terms (c^*) and (f^*)

$$\rho_{\infty}\widehat{v}\frac{\widehat{v}}{\lambda} \sim \mu\frac{\widehat{v}}{\lambda^2} \quad \text{or} \quad \frac{\widehat{v}}{U_{\infty}} \sim \frac{\mu}{\rho_{\infty}U_{\infty}\lambda} \sim \frac{D}{\lambda}\frac{1}{Re}$$
(26)

Using Eq. 6 or 25, Eq. 26 reproduces Eq. 9.

A4 Balancing terms (d^*) and (f^*)

$$\frac{\widehat{p}}{\overline{\lambda}} \sim \mu \frac{\widehat{v}}{\overline{\lambda^2}} \quad \text{or} \quad \widehat{p} \sim \mu \frac{\widehat{v}}{\overline{\lambda}} \quad \text{or} \quad \widehat{p} \sim \rho_\infty U_\infty \frac{\mu}{\rho_\infty U_\infty \lambda} \frac{\widehat{v}}{U_\infty}$$
(27)

and using Eq. 6 and 9 for large Re one obtains

$$\widehat{p} \sim \rho_{\infty} U_{\infty} \frac{\mu}{\rho_{\infty} U_{\infty} D} \frac{D}{\lambda} \frac{\widehat{v}}{U_{\infty}} \quad \text{or} \quad \widehat{p} \sim \rho_{\infty} U_{\infty} \frac{\mu}{\rho_{\infty} U_{\infty} D}$$
(28)

But recalling Eq. 13, Eq. 28 shows that term (f^*) may be neglected compared to (d^*) and thus the transverse gradient of pressure may be set to zero, i.e. scaling analysis has reproduced the classical boundary layer result. Note that term (e^*) is clearly small compared to term (f^*) .

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