Elastic and viscoelastic panel flutter in incompressible, subsonic and supersonic flows

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Abstract
An analytical investigation of elastic and viscoelastic panel flutter is undertaken. Through the analysis of an elementary linear homogeneous isotropic flat plate, it is shown that elastic or viscoelastic panel flutter in the form of simple harmonic motion is possible at incompressible to supersonic potential flow speeds. The low velocity range is of particular importance to UAVs, MAVs, wind turbines, helicopters, general aviation vehicles and fast flying vehicles during take-off/launch and landing. It is also shown that while the elastic / viscoelastic correspondence principle can be applied to linear plate (panel) formulations for stresses, strains and deformations, no such correspondence relations exist for elastic and viscoelastic panel flutter velocities and frequencies. The convergence of the Galerkin deflection series is investigated and its influence on flutter velocities and frequencies, and on plate deflections is evaluated. The influences of the panel spatial slopes in the airflow direction on flutter conditions are also analyzed and evaluated.

1. Introduction

The general topics of static and dynamic aeroelasticity are extensively covered in [74, 6, 5, 29, 47, 15, 80] among others. Flight vehicle panel flutter effects in flat and curved plates and in shells are of significant concern because of the likelihood of either immediate dynamic failures or long term material fatigue catastrophes. The current high interest in UAVs, MAVs and wind turbines requires consideration of the possibilities that these events will occur at low flight velocities (< 150 km/hr) in highly flexible light weight wings, tail surfaces, fuselages, ailerons, flaps, panels, blades, etc.

A considerable body of publications exists dealing with elastic panel flutter and is summarized in [20, 17, 50, 51, 18, 1, 31, 21, 23, 67, 24, 22] while aero-viscoelastic treatments are only emerging [32, 33, 10, 81, 61, 44, 72, 55, 25, 27, 26, 3, 2, 73, 39, 53, 75, 49, 4, 54, 71, 65, 66, 42, 43, 64, 63, 46, 62]. The detailed analysis in [19] raises the important question as to when the plate response is one of noise due to turbulent flow or when flutter in the form of simple harmonic motion (SHM) manifests itself. In Ref. [72] an analysis is presented for viscoelastic plate flutter subjected to random loads in a supersonic flow with the stationary loads in the form of Gaussian white noise. Probabilistic viscoelastic material stress-strain analysis and failure conditions have been treated in [41] and [40] respectively and important contributions of aerodynamic noise are described in [30]. In the present paper, the analysis is predicated on deterministic viscoelastic material properties and deterministic non-turbulent flow, and the conditions at low and/or high velocities under which possible elastic or viscoelastic panel SHM takes place are investigated.

Unsteady aerodynamic theory formulations are described in [6, 5, 20, 21, 23, 22, 9] to mention but a few.

In [69] the related, though distinct but more complicated, problem of subsonic axial flow in elastic thin walled cylinders is investigated. The results of

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three analytical approaches indicate the possibility of cylindrical flutter at positive frequencies. Experimental data reported in [28] for a plate on an elastic foundation indicates the presence of subsonic flow panel flutter in the form of either traveling or standing waves.

In the present paper it is shown by the use of a simple example consisting of a linear flat plate with 4 sides s.s. that elastic and viscoelastic panel flutter boundaries in the form of SHM can be achieved in potential flow at any speed – including incompressible flow – provided the proper mixtures of phase relations are realized between aerodynamic, inertia and structural forces. If the panel flutter problem is expanded to incorporate additional in-plane, thermal, control, piezoelectric, magneto-restrictive, smart material and/or other linear and nonlinear forces, the above conclusions remain the same but panel flutter will occur at different specific velocities and frequencies.

Panel flutter at low subsonic speeds is abundantly present and visible in nature in the form of flutter of awning panels, porch and window screens, flags, thin aluminum truck panels, UAVs, MAVs and wind turbine blades, among others.

Ultimately, the presence of system aeroelastic or aero-viscoelastic instabilities depends on the composite selection of parameters which directly influence the coefficients of the governing integro-differential relations and on the phase relations between the active forces.

In Refs. [65, 66, 42, 43] it is shown that panel flutter speeds can be significantly increased by respectively introducing aero-servo-viscoelastic controls or small amplitude pressure disturbances, and thus improving flutter conditions through postponement. Neither are considered in the present paper and only panel flutter boundaries (eigenvalues) are analyzed.

2. Analysis

2.1 Constitutive Relations

The investigation is carried out in a Cartesian coordinate system $x = \{x_i\}$ with $i = 1, 2, 3$ operating under the Einstein tensor notation rules. The system has a degenerate 2–D form, $\widetilde{x} = \{x_\alpha\}$ with $\alpha = 1, 2$ and the flow is in the x_1 direction with the x_3 coordinate normal to the panel. The panel material obeys linear isothermal, isotropic, homogeneous viscoelastic constitutive relations in terms of relaxation moduli E_{ijkl} [11] [34]

$$\begin{aligned}\sigma_{ij}(x,t) &= \int_{-\infty}^{t} E^{*}_{ijkl}(t-t')\ \epsilon_{kl}(x,t')\ dt' \\ &= \underbrace{E_{ijkl}(0)\ \epsilon_{kl}(x,t)}_{\text{instantaneous elastic response}} + \underbrace{\int_{0}^{t} E_{ijkl}(t-t')\ \frac{\partial \epsilon_{kl}(x,t')}{\partial t'}\ dt'}_{\text{time dependent viscoelastic response}} \end{aligned} \tag{1}$$

or conversely in terms of creep compliances C_{ijkl}

$$\begin{aligned}\epsilon_{ij}(x,t) &= \int_{-\infty}^{t} C^{*}_{ijkl}(t-t')\ \sigma_{kl}(x,t')\ dt' \\ &= \underbrace{C_{ijkl}(0)\ \sigma_{kl}(x,t)}_{\text{instantaneous elastic response}} + \underbrace{\int_{0}^{t} C_{ijkl}(t-t')\ \frac{\partial \sigma_{kl}(x,t')}{\partial t'}\ dt'}_{\text{time dependent viscoelastic response}} \end{aligned} \tag{2}$$

where the linear viscoelastic relaxation moduli are given by

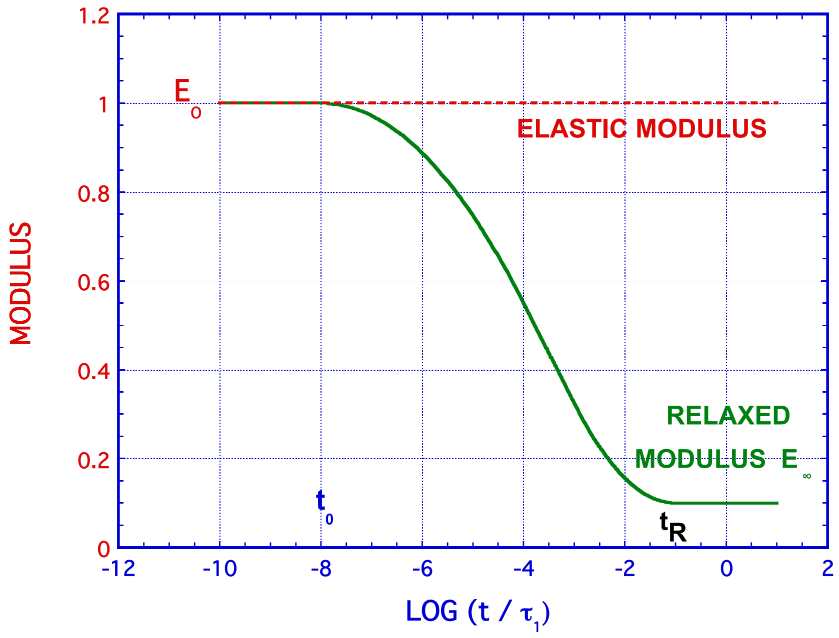

Figure 1: Elastic and viscoelastic moduli.

$$E_{ijkl}(t) \;=\; \overbrace{E_{ijkl\infty}}^{\substack{\text{fully relaxed}\\ \text{modulus} =\\ \text{E}_{\text{ijkl}}(\infty)}} \;+\; \overbrace{\sum_{n=1}^{N} E_{\underline{ijkln}} \exp\left(-\frac{t}{\tau_{\underline{ijkln}}}\right)}^{\substack{\text{creep/relaxation dissipative contributions}\\ \text{by Prony series [13]}}} \tag{3}$$

or

$$E_{ijkl}(t) \;=\; \underbrace{\overbrace{E_{ijkl0}}^{\substack{\text{elastic}\\ \text{modulus} =\\ \text{E}_{\text{ijkl}}(0)}}}_{\substack{\text{instantaneous}\\ \text{response}}} \;+\; \underbrace{\sum_{n=1}^{N} E_{\underline{ijkln}} \left[\exp\left(-\frac{t}{\tau_{\underline{ijkln}}}\right) - 1\right]}_{\text{creep/relaxation viscoelastic contributions}} \tag{4}$$

where underlined indices indicate no summations. The fully relaxed moduli $E_{ijkl\infty}$ are defined by (See Fig. 1.)

$$E_{ijkl\infty} \;=\; E_{ijkl0} - \sum_{n=1}^{N} E_{ijkln} \qquad \text{with} \qquad E_{ijkl0} \;>\; E_{ijkl\infty} \;\geq\; 0 \tag{5}$$

The dimensions of the various viscoelastic material parameters are displayed in Table 1 and the listed dimensions apply to symbols with and without subscripts n, hence $E_n = E_n^* \, \tau_n$, etc. Conditions are at rest in the interval $-\infty \leq t < 0$ and therefore $\sigma_{ij}(x,t) = \epsilon_{ij}(x,t) = 0$ holds in the negative time plane. Typical elastic and viscoelastic moduli curves are shown in Fig. 1 .

Eqs. (1) in their degenerate form include elastic materials as the 3–D isotropic, homogeneous and isothermal Hooke's law [76]

$$\sigma_{ij}^{E}(x,t) \;=\; E_{ijkl0} \, \epsilon_{kl}^{E}(x,t) \qquad \text{and} \qquad \epsilon_{ij}^{E}(x,t) \;=\; C_{ijkl0} \, \sigma_{kl}^{E}(x,t) \tag{6}$$

There are two isotropic and at most 21 anisotropic elastic $E_{ijkl0}(x)$ and relaxation moduli $E_{ijkl}(x,t)$. While elastic Poisson's ratios [70] are useful material descriptors, their viscoelastic counterparts are stress, stress history and time dependent and hence have no unique relations to material properties [35, 45, 36, 38, 52, 77, 78, 58] Consequently, viscoelastic constitutive relations must be written in terms of relaxation moduli or creep compliances without recourse to Poisson's ratios. Therefore, it follows that the expressions for viscoelastic bending rigidities D_{ijkl} and D_{ijkl}^* must also be devoid of Poisson's

Table 1: Dimensions

Parameters	Symbols	Dimensions (F = force, L = length, T = time)
Relaxation moduli	$E_{ijkln},\ E^T_{ijn}$	[F/L^2], [F/L^2]
	$E^*_{ijkln},\ E^{*T}_{ijn}$	[F/(L^{2}T)], [F/(L^{2}T)]
Compliances	$C_{ijkln},\ C^*_{ijkln}$	[L^2/F], [L^2/F T]
Relaxation times	τ_{ijkln}	[T]
Differential operators	$\boldsymbol{P}_{ij},\ \boldsymbol{Q}_{ijkl}$	[L/L], [F/L^2]
	$a_{ijn},\ b_{ijkln}$	[L T^n/L], [F T^n/L^2]
Bending rigidities	$D,\ D_{ijkln}$	[F L]
	$D^*,\ D^*_{ijkln}$	[F L/T]

ratios and that they must be formulated only in terms of moduli and plate geometries.

Alternately the isotropic homogeneous viscoelastic constitutive relations may be cast in a differential form, such that

$$\boldsymbol{P}_{\underline{ij}}\Big\{\sigma_{\underline{ij}}(x,t)\Big\} = \boldsymbol{Q}_{ijkl}\Big\{\epsilon_{kl}(x,t)\Big\} \tag{7}$$

with

$$\boldsymbol{P}_{ij} = \sum_{n=0}^{s} a_{ijn}\frac{\partial^n}{\partial t^n} \qquad \text{and} \qquad \boldsymbol{Q}_{ijkl} = \sum_{n=0}^{s} b_{ijkln}\frac{\partial^n}{\partial t^n} \tag{8}$$

The coefficients a_{ijn} and b_{ijkln} are material property parameters. The index $s = N$ of the Prony series (3) and (4) [34] Eqs. (1) and (2) are the solutions, i.e. Green's functions, of the differential relations (7). The isotropic elastic Hooke's law is given by the expressions

$$s = 0 \qquad \boldsymbol{P}^E_{ij} = a_{ij0} = 1 \qquad \boldsymbol{Q}^E_{ijkl} = b_{ijkl0} = E_{ijkl0} \tag{9}$$

and is contained in (7) above. After the elastic stress-strain relations, the simplest, i.e. most degenerate, example of viscoelastic constitutive relations is

$$s = 1 \quad \boldsymbol{P}_{ij} = a_{ij1}\frac{\partial}{\partial t} + \underbrace{a_{ij0}}_{=\,0} \quad \text{and} \quad \boldsymbol{Q}_{ijkl} = b_{ijkl1}\frac{\partial}{\partial t} + b_{ijkl0} \quad \text{with} \quad \frac{b_{\underline{ij}kl1}}{a_{\underline{ij}1}} = E_{ijkl0} \tag{10}$$

In general, the use of these differential expressions is awkward and somewhat impractical as real materials, i.e. high polymers, require r and s values of 25 to 30 for proper characterization. The integral formulations (1) are, therefore, the expressions of choice.

The possible presence of structural damping, i. e. Coulomb friction [12] necessitates that the terms E_{ijkl0} in (9) be altered to $(1 + \imath g_{\underline{ijkl}})E_{\underline{ijkl}0}$. The non-dimensional parameters g_{ijkl} are the coefficient of structural damping (generally $0 \leq g_{ijkl} \leq 0.05$) and are totally unrelated to the gravitational constant. Structural damping is due to friction in structural joints and is not a material property of the structural components, but rather is a manufacturing condition of the structural joints between panels and stringers and ribs.

2.2 Plate governing relations

As a relatively simple illustrative problem,[1] consider the isothermal dynamic equilibrium of a flat rectangular plate (panel) of dimensions[2] $a \times b \times h$ made of homogeneous isotropic linear elastic or viscoelastic materials. The air flow at a constant velocity V is in the x_1-direction. In the absence of in-plane tractions, thermal expansions and control forces, the governing relations simplify to and remain linear

$$\mathcal{L}(w) = \underbrace{\rho_{PL}\,\frac{\partial^2 w}{\partial t^2}}_{\text{inertia }(\mathrm{T}_1)} + \underbrace{\int_{-\infty}^{t} D^*(t-t')\;\nabla^4 w(\widetilde{x},t')\;dt'}_{\substack{\text{internal viscoelastic}\\ \text{bending resistance }(\mathrm{T}_2)}} + \underbrace{q_P\left(w,\;\overbrace{\frac{\partial w}{\partial x_1},\;\frac{\partial^2 w}{\partial x_1 \partial t}}^{\substack{\text{spatial derivatives}\\ \text{of interest}}},\;\frac{\partial w}{\partial t},\;\frac{\partial^2 w}{\partial t^2},\;\frac{\partial^2 w}{\partial x_1^2}\right)}_{\text{flexible panel aerodynamic pressure }(\mathrm{T}_3)} =$$

$$- \underbrace{q_L(\widetilde{x},t)}_{\substack{\text{rigid panel}\\ \text{aerodynamic}\\ \text{pressure }(\mathrm{T}_4)}} \qquad \text{with} \qquad \nabla^4 = \frac{\partial^4}{\partial x_1^4} + 2\frac{\partial^4}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4}{\partial x_2^4} \tag{11}$$

and where $w = w(\widetilde{x},t) = w(x_1,x_2,t)$ unless otherwise noted. The general isotropic viscoelastic bending rigidity is defined by

$$D_p^*(t) = \int_{-h/2}^{h/2} \underbrace{\widehat{D}_p^*\left[(E_{1111}(x_3,t), E_{1122}(x_3,t)\right]}_{\substack{\text{homogeneous or nonhomogeneous}\\ \text{and isotropic or anisotropic}}}\; x_3^2\,dx_3 = \underbrace{\frac{\widehat{D}^*(t)\,h^3}{12}}_{\substack{\text{homogeneous}\\ \text{isotropic plate}}} \qquad (p = 1,2,3) \tag{12}$$

If one integrates by parts, then the T_2 term can also be written as

$$\underbrace{\int_{-\infty}^{t} D^*(t-t')\;\nabla^4 w(\widetilde{x},t')\;dt'}_{\substack{\text{internal viscoelastic}\\ \text{bending resistance }(\mathrm{T}_2)}} = \underbrace{D_0\;\nabla^4 w(\widetilde{x},t)}_{\substack{\text{instantaneous elastic}\\ \text{response }(\mathrm{T}_{2\mathrm{E}})}} + \underbrace{\int_0^t \sum_{n=1}^{N} D_n \exp\left(\frac{t-t'}{\tau_n}\right)\frac{\partial \nabla^4 w(\widetilde{x},t')}{\partial t'}\,dt'}_{\text{time dependent viscoelastic response }(\mathrm{T}_{2\mathrm{V}})} \tag{13}$$

provided $w(x,t) = 0$ for $t < 0$. Consequently, the elastic and viscoelastic governing relations (11) differ only by the T_{4V} term.[3] However, the solutions $w^E(\widetilde{x},t)$ and $w(\widetilde{x},t)$ are equal only at $t = 0$. Should structural damping be active, then Term T_{2E} can be modified to read $(1 + \imath g)D_0$. The relations (11) and the more general ones of Appendix B can also be derived by applying the elastic-viscoelastic correspondence principle using Laplace or Fourier transforms.

In the case of linear aerodynamics, the aero-elastic/viscoelastic panel pressure is given by one or more terms of the type

$$\underbrace{q_P(\widetilde{x},t)}_{\text{Term }\mathrm{T}_3} = \underbrace{A_1\,w}_{\mathrm{T}_{3.1}} + \underbrace{A_2\,\frac{\partial w}{\partial x_1}}_{\mathrm{T}_{3.2}} + \underbrace{A_3\,\frac{\partial w}{\partial t}}_{\mathrm{T}_{3.3}} + \underbrace{A_4\,\frac{\partial^2 w}{\partial x_1 \partial t}}_{\mathrm{T}_{3.4}} + \underbrace{A_5\,\frac{\partial^2 w}{\partial t^2}}_{\mathrm{T}_{3.5}} + \underbrace{A_6\,\frac{\partial^2 w}{\partial x_1^2}}_{\mathrm{T}_{3.6}} \tag{14}$$

[1]More complicated examples include anisotropic nonhomogeneous auxetic materials, nonlinear and/or temperature effects, in-plane tractions, control forces, nonlinear aerodynamics and materials, large deformations, random loads and/or material properties, aerodynamic noise, turbulence, curved plates, shells, etc. See Eq. (45) in Appendix B.

[2]The x_1-dimension a has no relation to the material property parameters a_n defined in (8).

[3]If all D_n are set to zero for $1 \le n \le N$, then the elastic solution $w^E(\widetilde{x},t) = w(\widetilde{x},t)$ for $0 \le t \le \infty$ emerges. See Appendix B for definitions of the bending rigidities D.

with the coefficients $A_r = A_r(V, \omega)$ and $r = 1, 2, \cdots, 6$, defined by the sub-, trans- or super- sonic aerodynamics. V [L/T] is the flight velocity and ω [1/T] is the frequency. (See Appendix A for some examples of these aerodynamic coefficients.) In each case, the flexible body aerodynamics change the character of the governing relations by altering the order of the spatial and/or temporal deflection derivatives, and the coefficients multiplying these derivatives. *However, that does not redefine the nature of the time exponential form of the solution (18), only the specific values of the flutter eigenvelocities and eigenfrequencies and of the phase relations are altered.*

Similarly, the rigid body pressure (lift) is given by

$$q_L(\widetilde{x}, t) \;=\; A_0\left(\rho, V, \alpha, \frac{dC_L}{d\alpha}, \text{airfoil geometry}\right) \tag{15}$$

where ρ is the air density and α the rigid body angle of attack.

Alternately by using the differential isotropic stress-strain relation form (8), Eq. (11) may be written as[4]

$$\begin{aligned}\mathcal{L}_1(w) \;=\; & \underbrace{\boldsymbol{P}\left\{\rho_{PL}\,\frac{\partial^2 w}{\partial t^2}\right\}}_{\text{inertia force }(\mathrm{T_{1D}})} \;+\; \underbrace{\frac{h^3}{12}\boldsymbol{Q}\left\{\nabla^4 w\right\}}_{\substack{\text{internal viscoelastic}\\ \text{bending resistance }(\mathrm{T_{2D}})}} \\ & + \underbrace{\boldsymbol{P}\left\{q_P\left(w, \overbrace{\frac{\partial w}{\partial x_1}, \frac{\partial^2 w}{\partial x_1 \partial t}}^{\substack{\text{spatial derivatives}\\ \text{of interest}}}, \frac{\partial w}{\partial t}, \frac{\partial^2 w}{\partial t^2}, \frac{\partial^2 w}{\partial x_1^2}\right)\right\}}_{\text{flexible panel aerodynamic pressure }(\mathrm{T_{3D}})} \;=\; -\,\underbrace{\boldsymbol{P}\Big\{q_L(\widetilde{x}, t)\Big\}}_{\substack{\text{rigid panel}\\ \text{aerodynamic}\\ \text{pressure }(\mathrm{T_{4D}})}}\end{aligned} \tag{16}$$

which perhaps offers a clearer insight to the stability of the deflection $w(\widetilde{x}, t)$ than the integral governing relations (11). With the operators defined by Eqs. (9), the highest time derivative of this partial differential equation (PDE) is of the order $s+2$ and its coefficients depend on a blend of elastic moduli and/or viscoelastic material derivatives, density, bending rigidities and aerodynamic factors. The interrelations among the coefficients of these PDE derivatives determine the panel system stability.

The solutions of the homogeneous parts of Eqs. (11) and (16),

$$\mathcal{L} \;=\; \mathcal{L}_1 \;=\; 0 \tag{17}$$

are of the form

$$w(\widetilde{x}, t) \;=\; \sum_{m=1}^{8} B^*_m(V, \omega)\exp\left[\left(\widehat{d}_m + \imath\,\omega_m\right)t\right]\, W_m\left(\widetilde{x}\right) \tag{18}$$

with each and every $W_m(\widetilde{x})$ function satisfying the BCs. The stability of the

[4]Note that the numerical subscripts identifying T_i terms are identical with those of the integro-differential governing relation (11). See also Eq. (45).

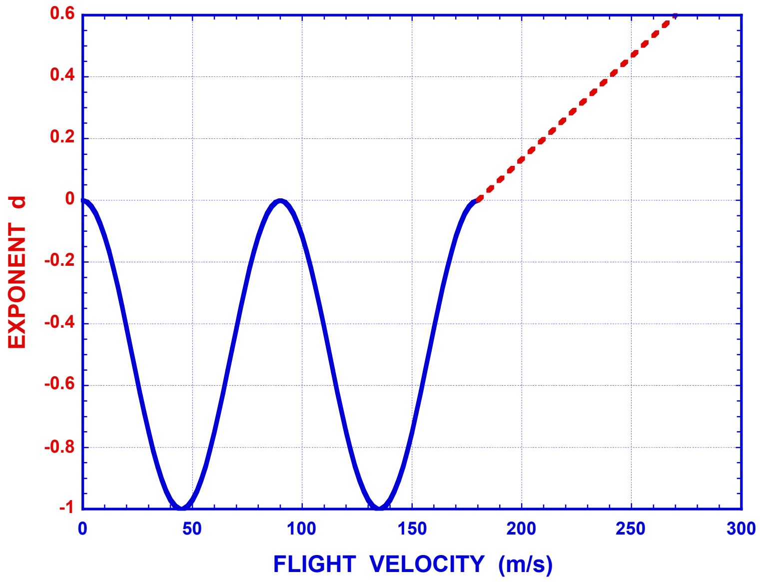

Figure 2: Variation of exponent $\widehat{d}$ with flight velocity, Eqs. (20) and (21).

plate is defined by the parameter $\widehat{d}$ as

$$\widehat{d}_m\Big(V,\omega,E_n,\tau_n,\rho_{PL},\rho,\frac{dC_L}{d\alpha},\text{geometry}\Big) \Longrightarrow \begin{cases} <\ 0 \ \stackrel{\text{def}}{\Rightarrow}\ \textit{dynamically stable},\ \lim_{t\to\infty}\Big\{u(x,t)\Big\}\to 0 \\ =\ 0 \ \stackrel{\text{def}}{\Rightarrow}\ \textit{neutrally stable at } V=V_f \text{ and } \omega=\omega_f, \\ \quad \text{i. e. SHM for } u(x,t), \text{ the legacy} \\ \quad \text{definition of } \textit{panel flutter} \text{ onset} \\ >\ 0 \ \stackrel{\text{def}}{\Rightarrow}\ \textit{dynamically unstable}, \text{ for } V>V_f \\ \quad \text{and } \lim_{t\to t_f}\Big\{u(x,t)\Big\}\to\infty \\ \quad \text{or } \lim_{t\to t_{f1}}\Big\{\frac{\partial u(x,t)}{\partial t}\Big\}\to\infty \\ <\ 0 \ \stackrel{\text{def}}{\Rightarrow}\ \textit{primarily time dependent material} \\ \quad \textit{and/or structural failure(s)} \\ \quad \text{(before dynamic instability predominates)} \\ \quad \lim_{t\to t_{ult}}\Big\{u_{max}(x,t)\Big\}\to u_{ult}(t_{ult}) \text{ or} \\ \quad \lim_{t\to t_{ult}}\Big\{\sigma_{max}(x,t)\Big\}\to\sigma_{ult}(t_{ult}) \end{cases} \tag{19}$$

with $0 < t_f \leq \infty$. The last condition, even though driven by aero-viscoelastic coupled forces, us a "simple" event where a prescribed failure condition has been violated.

In reality, the above neutrally stable condition only defines the onset of instability provided

$$\left.\frac{\partial\widehat{d}}{\partial V}\right|_{V=V_f} \neq 0 \tag{20}$$

If Eq. (20) is not satisfied then $\widehat{d}$ continues negative for increasing V, unless $\widehat{d}(V,\cdots)=0$ is an inflection point, when the constraints

$$\left.\frac{\partial\widehat{d}}{\partial V}\right|_{V=V_f} = \left.\frac{\partial^2\widehat{d}}{\partial V^2}\right|_{V=V_f} = 0 \quad\text{and}\quad \left.\frac{\partial^3\widehat{d}}{\partial V^3}\right|_{V=V_f} \neq 0 \tag{21}$$

must be fulfilled (See Fig. 2). The analysis of and the associated problems generated by starting transients are presented in detail in [64]

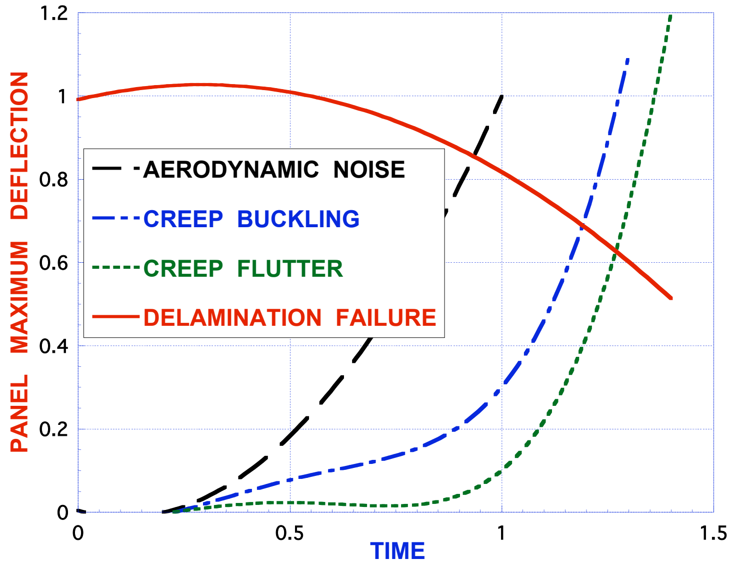

Figure 3: Viscoelastic panel responses.

Of course, the SHM due to the pressure q_P, Term T_3 in Eq. (11), represents panel excursions $w(\widehat{x}, t)$ in addition to the rigid body displacements produced by the the aerodynamic pressure q_L of Term T_4. Additionally, both rigid and flexible wing bending and twisting deformations are present in the overall lifting surface system.

The panel flutter velocity and frequency are found in the "usual elastic manner" by satisfying eight homogeneous boundary conditions and setting the complex determinant of coefficients B_m equal to zero. The pair of eigenvalues containing the lowest value of the flight velocity $V = V_f > 0$ and its corresponding frequency $\omega_f > 0$ establish the flutter conditions of the panel.

Parenthetically, one needs to add that other forms of instabilities such as panel creep buckling and that outright material failures such as delaminations may occur at velocities unrelated to the flutter velocities. The latter may be larger or smaller than V_f. Fig. 3 is a schematic representation of these possibilities in relation to plate bending deflections indicating the concept of lifetime or survival time for each of these independent conditions.

Note that homogeneous partial (PDE) or integro-partial differential (IPDE) relations and homogeneous BCs do not yield any information about the displacement amplitudes B_m. On the other hand if one or more of the following are included in the governing relations

- the rigid body aerodynamic pressures (T_4) and/or
- initial imperfections $w_0(\widetilde{x})$ and/or
- at least one non vanishing BC

then expressions for the amplitudes are achievable but the eigenvalues V_f and ω_f may be obtainable only from the homogeneous parts of the governing PDEs or IDEs.

The homogeneous elastic formulation for Eq. (11) can also be realized by replacing $D(t)$ with

$$D^E = D_0 = \frac{E_0 \, h^3}{12\,(1 - \nu_0^2)} \tag{22}$$

and by eliminating the time integral in term T_{2V} in (13). It is possible to establish a correspondence between elastic and viscoelastic stresses, strains and displacements through the well established integral transform correspondence principle [11, 34]

Since the elastic-viscoelastic analogy specifically excludes Poisson's ratios and is limited to expressions involving only moduli, compliances, creep and relaxation functions and convolution type constitutive relations [35, 45, 36, 42, 52, 77, 78, 58] it follows that in the Fourier transform plane

$$\overline{\overline{D}}(\Omega) = \underbrace{\int_{-\infty}^{\infty} \exp(-\imath\,\Omega\,t) D(t)\,dt}_{\text{Fourier transform (FT) of D(t)}} \neq \frac{\overline{\overline{E}}(\Omega)\,h^3}{12\left(1-\overline{\overline{\nu^2}}(\Omega)\right)} \tag{23}$$

Therefore, the transformed $\overline{\overline{D}}(\Omega)$ must be defined solely in terms of moduli as seen in Eq. (12).

However, no known relation exists between elastic (V_f^E) and viscoelastic (V_f) flutter velocities. This is due to the extremely complicated transcendental equations for $\widehat{d} = 0$ establishing the eigenvalues for which only numerical, rather than analytical, solutions can be realized.

The initial conditions of the viscoelastic panel are those of the equivalent elastic one and, therefore, at $t = 0$ the viscoelastic V_f must be less than the elastic V_f^E, provided the panel reaches SHM instantaneously. Once time increases and the viscoelastic relaxation moduli begin to decrease there no longer exist any such constraints on[5] V_f. Consequently, as determined by the PDE or IPD coefficients, the actual flutter velocities for $t > 0$ can be $V_f \gtreqless V_f^E$ depending on phase relationships between inertia, viscoelastic and aerodynamic forces [32, 66] (See more detail in Discussion Section.)

2.3 Boundary Conditions

For a simply supported plate on four sides (4 s.s.s.) with the coordinate system origin at $x_i = 0$ in the plate lower left corner, the BCs are

$$w\,(0, x_2, t) = w\,(a, x_2, t) = w\,(x_1, 0, t) = w\,(x_1, b, t) = 0 \tag{24}$$

and

$$\int_{-\infty}^{t} D^*(t-t')\,\frac{\partial^2 w(x_1, 0\,\text{or}\,b, t')}{\partial x_1^2}\,dt' = \int_{-\infty}^{t} D^*(t-t')\,\frac{\partial^2 w(0\,\text{or}\,a, x_2, t')}{\partial x_2^2}\,dt' = 0 \tag{25}$$

which turns Eq. (18) into the simple harmonic motion solution form (4 s.s.s.)

$$w(\widetilde{x}, t) = \exp\,(\imath\,\omega\,t)\,B_{11}(V,\omega)\,\sin\left(\frac{\pi\,x_1}{a}\right)\,\sin\left(\frac{\pi\,x_2}{b}\right) \tag{26}$$

This represents the first term of the general solution for 4 s.s.s.

$$w(\widetilde{x}, t) = \exp\,(\imath\,\omega\,t)\sum_{m=1}^{M^*\to\infty} B_{m1}(V,\omega)\,\sin\left(\frac{m\,\pi\,x_1}{a}\right)\,\sin\left(\frac{\pi\,x_2}{b}\right) \tag{27}$$

Depending on other BCs, combinations of circular and hyperbolic sine and cosine functions are also permissible, as well as any other suitable functions. For practical purposes, a truncated series is chosen and "convergence" is established through an examination of $w(\widetilde{x}, t)$ values with the addition of terms to the series until values of the function stabilize. (See Discussion Section and Tables 3 and 4.)

When (26) is substituted into the homogeneous part of the governing relation (11) or (16), i. e. Eq. (17), it leads to

$$\left[R_{R11}(V,\omega) + \imath\,R_{I11}(V,\omega)\right]\,\{B_1\} = 0 \tag{28}$$

[5] Any possible ambiguity about the ICs can be removed if one additionally imposes $w(x, 0) = w^E(x, 0) = 0$. However, neither vanishing or non-vanishing initial panel deflections affect the paired eigenvalues V_f and ω_f.

The paired eigenvalues V and ω are obtained by simultaneous solution of the following two relations

$$R_{R11}(V,\omega) \;=\; 0 \quad \text{and} \quad R_{I11}(V,\omega) \;=\; 0 \tag{29}$$

since the trivial solution $B_1 = 0$ is excluded.

For more complicated BCs involving up to eight non-zero B_{mn}s, Eq. (29) becomes the determinant

$$\left| R_{Rkl}(V,\omega) \;+\; \imath\, R_{Ikl}(V,\omega) \right| \;=\; 0 \qquad k,l = 1,2,3,4 \tag{30}$$

resulting in complex relations

$$\boldsymbol{R}(V,\omega) = 0 \tag{31}$$

or

$$\boldsymbol{R}_R(V,\omega) \;=\; 0 \quad \text{and} \quad \boldsymbol{R}_I(V,\omega) \;=\; 0 \tag{32}$$

In both the elastic and viscoelastic panel, the flutter velocity V_f is the lowest value of the real positive eigenvalues of V paired with a real $\omega_f > 0$. This couple can be found by trial and error or from nonlinear transcendental equation solvers or from the solution of the relations

$$\frac{\partial \boldsymbol{R}(V,\omega)}{\partial V} = 0 \quad \text{and} \quad \frac{\partial \boldsymbol{R}(V,\omega)}{\partial \omega} = 0 \quad \text{with} \quad V_f,\ \omega_f > 0 \quad \text{minimum values} \tag{33}$$

It is to be noted that conceptually Eqs. (30) to (33) and their solution protocols for the paired eigenvalues V and ω are identical for elastic and viscoelastic materials. However, specific values V_f and ω_f for the viscoelastic plate, and V_f^E and ω_f^E for the corresponding equivalent[6] elastic one will differ.

3. Discussion

3.1 Some mathematical proofs regarding subsonic panel divergence and flutter using Galerkin's method

Galerkin's method [48] has been used to find the approximate flutter solutions for rectangular panels. However, care must be exercised in selecting the number of terms to be used in the Galerkin series because of the inherent convergence properties of the method. An example is given in [56] dealing with static and dynamic instability of panels in subsonic potential flow. The paper attempts to prove that panels will only diverge at subsonic speeds and that flutter will not occur prior to divergence. The mathematical analysis itself is entirely correct and does demonstrate that subsonic divergence occurs before flutter, but one of the starting assumptions is problematic. In [56] it is mentioned that the series approximation is an infinite series "but by virtue of the convergence of Galerkin's method (e.g. see [59]) this can be ignored and the discussion confined to $j, m = 1, 2$ only." In the quotation j and m are the indices for the series expansion and, therefore, the referenced expansion is limited to only two terms. By limiting the expansion to two terms, the assumption is made that the results from a two term expansion are representative of the results from larger expansions.

Unfortunately, the above assumption is not correct, and it has been demonstrated in [60] that, for some cases, Galerkin's method can have a non-uniform convergence. In [60] two situations are examined, namely supersonic membrane flutter and a clamped-clamped rod with a follower force. The membrane flutter example originated in [7] where a paradox is identified, which states that Galerkin's method would produce a supersonic flutter velocity while the exact solution would not. This discrepancy is explained by demonstrating that membranes do not have a normal determinant, where a normal determinant is defined

[6] same plate BCs, ICs, geometry, E_0, etc.

by when the series $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|A_{ij}|$ converges. The A_{ij} represent the terms of the coefficient matrix derived from the system of governing equations. A typical plate problem in solid mechanics does have a normal determinant and, therefore, Galerkin's method produces reliable results. In [8] it is suggested that as a plate approaches a membrane, the convergence of Galerkin's method becomes slower to the point of failure when a membrane is actually reached. In [60] it is demonstrated that the discrepancy is the result of using normal determinants in the analysis of [8] If the more general conditions for the convergence of an infinite series by von Koch [79] are used, then Galerkin's method can also be shown to converge for the membrane flutter; however, the convergence is non-uniform in general. The three conditions (series) demonstrated by von Koch that must converge are

$$\sum_{i=1}^{\infty}|A_{ij}-1| \qquad \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|A_{ij}-1|^2 \qquad \sum_{i=1}^{\infty}x_i^2 \tag{34}$$

where $[A]$ is the coefficient matrix as before and $\{x\}$ is the vector of system variables that satisfy $[A]\{x\}=0$.

The non-uniform convergence of Galerkin's method is numerically demonstrated in the second example of a clamped-clamped rod with a follower force in [60] In [59] it was shown earlier that the rod diverged (buckled) with a two term expansion in the Galerkin method. However, if the expansion is taken to four terms, the rod is no longer buckled. In [8] it was demonstrated that a similar result can be obtained for supersonic membrane flutter where eight terms are required to demonstrate flutter.

While the membrane paradox originated with supersonic membranes, the subsequent non-uniform convergence of Galerkin's method shown in [60] appears to hold for subsonic panel flutter cases as shown by the present paper. It is suggested that to correctly use Galerkin's method for any panel flutter case that the von Koch conditions [79] are used to estimate the number of terms required for convergence.

3.2 General considerations

Circumstances leading to panel flutter are found in self-excited closed loop systems and are due to the interdependence between aerodynamic and inertia forces and structural deflections as well as their spatial and temporal derivatives. The one open flutter driver parameter in flight vehicles is their velocity. Consequently, for other fixed conditions[7] the lowest velocity, i. e. the flutter velocity, needs to be established for entire vehicles and for their components such as outer skin panels. In the present linear formulation, the existence of SHM is controlled by the relative values of the coefficients of displacements and their derivatives in Eqs. (17), which in turn are governed by the interrelations between inertia, elastic/viscoelastic, aerodynamic, thermal, in-plane and control forces. Material properties determine Term T_2, while the type of aerodynamic flow and the flight velocity V control the nature of of Term T_3 in (11) and/or (16). Additionally, the BCs directly affect the matrix of the coefficients of the B_m amplitudes of (18). All of the above conditions coalesce to determine the eigenvalues and hence the flutter conditions V_f and ω_f.

In Eq. (45) of Appendix B, the in-plane tractions T_5, T_7, T_9 and T_{11} due to deflections are seen to be inherently nonlinear, while the aerodynamic lift terms T_{13} can be linearized for small angles ($\lessapprox 0.1$ rad) by exchanging angular arguments for the sine and arctangent. It was demonstrated in Ref. [37] that for steady state temperatures, the thermal loads T_{16} to T_{18} not only change their values in time but may even reverse signs, thus changing tensile to compressive

[7]aerodynamic shapes, mass distributions, material properties, controls, trim angles of attack, etc.

tractions and vice versa. Additionally, in steady state elasticity, in-plane tensions generally tend to stabilize the system, unfortunately the same cannot be said for equivalent[6] quasi-steady state viscoelastic systems because the phase relations are different for the elastic and viscoelastic governing relations. Stability conditions must be examined on a case by case basis.

The presence of viscoelastic material damping, or for that matter external and/or structural damping – Terms T_2 and T_3 in (45) – offers no guarantees that viscoelastic flutter speeds V_f will be higher or lower than the corresponding[6] elastic V_f^E. The values of these V_fs can move in either direction depending again on the character of the phase relationships between aerodynamic, elastic/viscoelastic and inertia forces. Similar comments can be made regarding the contributions made by changes in mass and aerodynamic forces. However, since Eqs. (28) and (30) are transcendental relations in the eigenvalues V_f and ω_f no analytical solutions are attainable and hence no general conclusions about the interrelations, if any, between elastic and viscoelastic panel flutter velocities can be drawn.

Of course, in addition to or instead of flutter the panel may also experience creep buckling and/or failures such as delamination, crack propagation, fatigue, etc. [66] which will limit the panel's lifetime, but neither phenomenon is considered here as the concentration of this study is solely on panel flutter. Depending on the phase relations any one of the three phenomena – panel flutter, creep buckling, material failures – can precede the other two in time and, therefore, make it the dominant analysis/design criterion of necessity. In Fig. 3 the various possible panel modes of deflection and survival times are illustrated. The relative time wise position of each curve is dependent on individual plate parameters, such as relaxation moduli, inertia, aerodynamic forces, etc., and will vary accordingly.

For purposes of general stability discussions, consider the simplest elastic and viscoelastic examples governed by constitutive equations (9) and (10). Applying Galerkin's method [48] to (16) removes the x-dependences[8] reduces the relations to time ODEs of the type[9]

$$\text{elastic} \implies A_2^E \ddot{\mathcal{W}}^E(t) + A_1^E \dot{\mathcal{W}}^E(t) + A_0^E \mathcal{W}^E(t) + A_{00}^E = 0 \tag{35}$$

$$\text{simple viscoelastic}^{10} \implies A_3^V \dddot{\mathcal{W}}(t) + A_2^V \ddot{\mathcal{W}}(t) + A_1^V \dot{\mathcal{W}}(t) + A_0^V \mathcal{W}(t) + A_{00} = 0 \tag{36}$$

The A_k coefficients contain portions which are independent of aerodynamic input (T_1, T_2, Eq. (11)) and portions that are functions of the flight velocity V ($\mathrm{T}_{3.1}$ to $\mathrm{T}_{3.6}$, Eq. (14)). See Table 2.

For the general viscoelastic stress-strain relations, Eq. (36) becomes

$$\text{general viscoelastic} \implies A_2^V \ddot{\mathcal{W}}(t) + A_1^V \dot{\mathcal{W}}(t) + A_0^V \mathcal{W}(t)$$

$$+ A_{INT}^V \left[\underbrace{E_0\, \mathcal{W}(t)}_{\substack{\text{elastic}\\ \text{contribution}}} + \underbrace{\int_0^t E(t-t')\, \dot{\mathcal{W}}(t')\, dt'}_{\text{creep / relaxation contribution}} \right] + A_{00} = 0 \tag{37}$$

[8]Alternately, one can employ separation of variables to extract the time ODE portions of the governing relations or substitute the solution form (18) that satisfies the BCs, and then manipulate the coefficients in the same manner as outlined above for the Galerkin approach.

[9]The application of Galerkin's method to a flexible viscoelastic panel on "rigid" supports creates the physical equivalent of a rigid plate on flexible viscoelastic supports.

[10]Based on first derivatives only stemming from the alternate differential formulation of the constitutive relations, Eqs. (7).

ODE Term	Contributing Terms	Source of Terms	Eq. No.
A_2^E	T_1, $T_{3.5}$	inertia & aerodynamics	elastic
A_1^E	$T_{3.3}$, $T_{3.4}$	aerodynamics	(35)
A_0^E	T_2, $T_{3.1}$, $T_{3.2}$, $T_{3.6}$	stiffness & aerodynamics	
A_3^V	T_1, $T_{3.5}$	inertia & aerodynamics	simple
A_2^V	$T_{3.3}$, $T_{3.4}$	aerodynamics	viscoelastic
A_1^V	T_2, $T_{3.1}$, $T_{3.2}$, $T_{3.6}$	stiffness & aerodynamics	(36)
A_0^V	T_2	stiffness	
A_2^V	T_1, $T_{3.5}$	inertia & aerodynamics	general
A_1^V	$T_{3.3}$, $T_{3.4}$	aerodynamics	viscoelastic
A_0^V	T_{2E}, $T_{3.1}$, $T_{3.2}$, $T_{3.6}$	elastic stiffness (13) and aerodynamics	(37)
A_{INT}^V	T_{2V}	creep stiffness (13)	

Table 2: Aerodynamic influence on Eqs. (35) to (37) coefficients.

Next one substitutes algebraic variables for the derivatives according to $\mathcal{U}^n = d^n\mathcal{W}/dt^n$. Application of an aerodynamic theory appropriate to each flight regime allows determinations of each coefficient A_k as functions of V. For $V = 0$ all coefficients in both relations will be positive. This is due to the fact that with vanishing Vs these remaining coefficients represent only inertia, material and structural open loop contributions.

Applying Descartes' rule of signs [14] to the modified homogeneous portions of the PDEs, i. e.

$$\text{elastic} \implies A_2^E\,(\mathcal{U}^E)^2 \;+\; A_1^E\,\mathcal{U}^E \;+\; A_0^E \;=\; 0 \tag{38}$$

$$\text{simple viscoelastic} \implies A_3^V\,\mathcal{U}^3 \;+\; A_2^V\,\mathcal{U}^2 \;+\; A_1^V\,\mathcal{U} \;+\; A_0^V \;=\; 0 \tag{39}$$

determines the number of possible positive real roots $\widehat{d}$ defining the solution (See (18))

$$\mathcal{W}(t) \;\sim\; exp\Big[\Big(\widehat{d} + \imath\,\omega\Big)\,t\Big] \tag{40}$$

Therefore, when $V = 0$ there are no real positive roots and $\widehat{d} \leq 0$ indicating stability of elastic and simple viscoelastic (36) motions. However, Descartes' rule does not account for any possible positive parts of the complex roots $\widehat{d}$, which may yield larger or smaller V_fs than those emerging from the positive real roots. Furthermore, Descartes' rule cannot be applied to the general viscoelastic integro-differential relation (37). However it can be used in conjunction with the general PDE (16).

At this point a caveat needs to be introduced when the selected $W(\widetilde{x})$s of Eq. (18) are orthogonal functions. The latter impinges on the aerodynamic term

$T_{3.2}$ and $T_{3.4}$ contributions which are out of phase with all others. If integral methods are applied, such as for instance Galerkin's approach, the orthogonality may remove these aerodynamic contributions either partially or totally. This is indeed the case in [17] and [21] and in this paper, Eq. (26), when only a single sine or cosine term is used to define the x_1 dependence of $w(\widetilde{x}, t)$. However, if more than on term is chosen then the influences of the two aerodynamic terms generally remain after the Galerkin application.

Two seemingly inexplicable questions remain:

- (A) What is the relation, if any, between the error introduced by assuming an approximate form for the plate deflection $w(\widetilde{x}, t)$, as in (26), and the resulting imprecise eigenvalues for the flutter V_f and ω_f pair? – No such expression appears derivable.

- (B) How large an error can be tolerated in the approximate $w(\widetilde{x}, t)$ expression before the associated resulting inexact V_f value can be trusted? – In the absence of an analytic relation between V_f and w, the V_f convergence and error analyses have to be established independently of the $w(\widetilde{x}, t)$ convergence. (See Tables 3 and 4.)

The coefficients of the governing relations are special conglomerates of inertia, aerodynamic and bending stiffness contributions. The solution and stability of these relations depend only on the fortuitous relative values of their coefficients and not on where their contributions may originate. The panel aerodynamic pressure (T_3 terms in (14)) is defined in terms of deformations and their spatial and temporal derivatives. The presence or absence of any one or more aerodynamic terms is not of primary contributory importance as their missing values can be compensated for by adjusting other parameters in the coefficient group, except for the combination $T_{3.3}$ and $T_{3.4}$ which is purely aerodynamic. (See Table 2.)

In the final analysis, it must be remembered that the governing relations used here or elsewhere in similar or different forms, whether linear or nonlinear and more or less sophisticated, represent but a model and not necessarily the real world.[11] The behavior of these relations and their stability are unique to the chosen model and their associated parameters, and may or may not approximate or accurately simulate reality in every or most details.

3.3 A few illustrative examples

In the isothermal plate illustrative examples no additional thermal moments or in-plane forces are introduced. Nor are any external control forces/moments applied to the example panel. The latter preserve the linearity of the problem – see Eq. (45). Representative results of a some illustrative examples are displayed in Tables 3 to 5 and in Figs. 4 to 6. The dimensions of the current four sides simply supported and fixed panels are $a = b = 0.1$ m, thickness $h = .0005$ m with density $= 2700$ kg/m^3 and an air density of 1.225 kg/m^3. The Galerkin integrals were evaluated analytically on a laptop using MATLAB™ symbolic routines.

Tables 3 to 5 summarize elastic and viscoelastic results. Increases in τ values are due to decreased temperatures, i. e. increases in temperatures lead to faster creep rates and smaller relaxation times. The entries with $\tau = 0$ are elastic panels, while the others with $\tau \neq 0$ represent viscoelastic materials. Note that all examined cases for the selected parameters produce flutter velocities and frequencies at reasonable positive values. No nearly zero Hz frequencies were encountered, even though the iterative elastic eigenvalues solutions were deliberately started at $\omega = 0$. This is unlike the results obtained in [18] and [21]

[11] A parallel exists in semantics theory defined among others by the statement: "The map is not the territory" [57]

Table 3: Galerkin approach results for an elastic four S.S.S. panel exposed to subsonic potential flow with / without x_1 Derivatives.

No. of Terms M^* in Eq. (27) with $N^* = 1$	V_f (m/s)	ω_f (Hz)	w_{max} (m/m)	Convergence V_f	Convergence ω_f	Convergence w_{max}
1	85 / 85	6.13 / 6.13	0.000 / 1.000			
2	85 / 85	5.23 / 6.13	0.951 / 1.000	Y/Y		
3	85 / 85	5.13 / 6.06	0.951 / 1.004			
4	85 / 85	5.13 / 6.06	1.006 / 1.004		Y/Y	N/Y
6	85 / 85	5.13 / 6.06	0.956 / 1.004			
8	85 / 85	5.13 / 6.06	0.956 / 1.004			Y/Y
10	85 / 85	5.13 / 6.06	0.956 / 1.004			

Table 4: Galerkin approach results for an elastic four S.S.S. panel exposed to supersonic potential flow with / without x_1 derivatives.

No. of Terms M^* in (27) with $N^* = 1$	V_f (m/s)	ω_f (Hz)	w_{max} (m/m)	Convergence V_f	Convergence ω_f	Convergence w_{max}
1	485 / 350	24.77 / 24.77	0.000 / 1.000			
2	980 / 975	45.49 / 44.91	1.244 / 1.242			
3	1655 / 1615	56.76 / 55.53	1.334 / 1.324			
4	1380 / 1365	52.46 / 51.76	1.300 / 1.295			
6	1405 / 1390	52.88 / 51.95	1.300 / 1.291			Y/N
8	1410 / 1390	52.96 / 51.62	1.300 / 1.284	N/Y		
10	1410 / 1390	52.96 / 51.57	1.300 / 1.284	Y/Y	Y/N	Y/Y

where almost zero frequencies were reported for their 2 sides s.s and 2 sides free elastic subsonic panels.

In the previous publications [18, 1, 31, 21] which indicate lack of subsonic panel flutter, the results appear to have come from a premature truncating of the deflection series. The convergence of the series (27) and of the corresponding flutter conditions were investigated by including successively more terms in the $w(\widetilde{x}, t)$ series until the solutions "converged" as shown in Tables 3 and 4. Note the radical changes in flutter frequency values between one and two term series solutions and the different convergence occurrences for V_f, ω_f and w_{max}.

3.4 Low and high speed elastic panel flutter

Tables 3 and 4 display results for sub- and super- sonic illustrative examples with typical ordinary properties. Both cases with x_1 derivatives present and absent in the aerodynamic force definition of Eq. (14) are considered. The effect of terms $T_{3.2}$ and $T_{3.4}$ seems to be negligible on the flutter velocity and greatest on both the flutter frequency and bending deflection.

The possible presence of these first spatial derivatives in some of the aerodynamic force definitions[12] radically alters the character of the plate governing relations (11) and (45). Without either or both of these terms, in addition to time derivatives only derivatives of even orders in x_1 and the function itself are present in the governing relations. For instance, the illustrative example of 4 s.s.s. BCs requires only sine terms in the series for the deflection $w(\widetilde{x}, t)$, Eqs. (18) and (27), to satisfy the BCs. Without terms $T_{3.2}$ and $T_{3.4}$ these expressions also satisfy the aforementioned governing relations term by term.

[12]Terms $T_{3.2}$ and/or $T_{3.4}$ in Eq. (14)

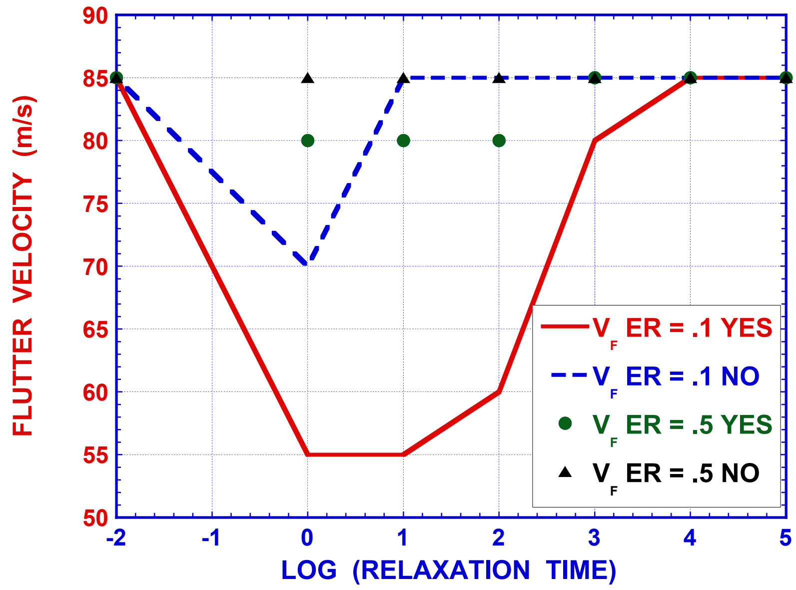

Figure 4: Subsonic panel flutter velocity.

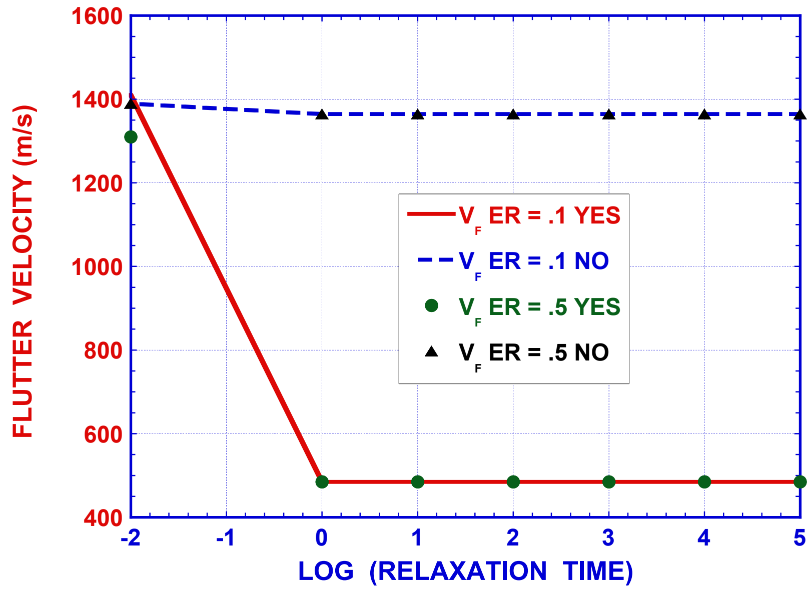

Figure 5: Supersonic panel flutter velocity.

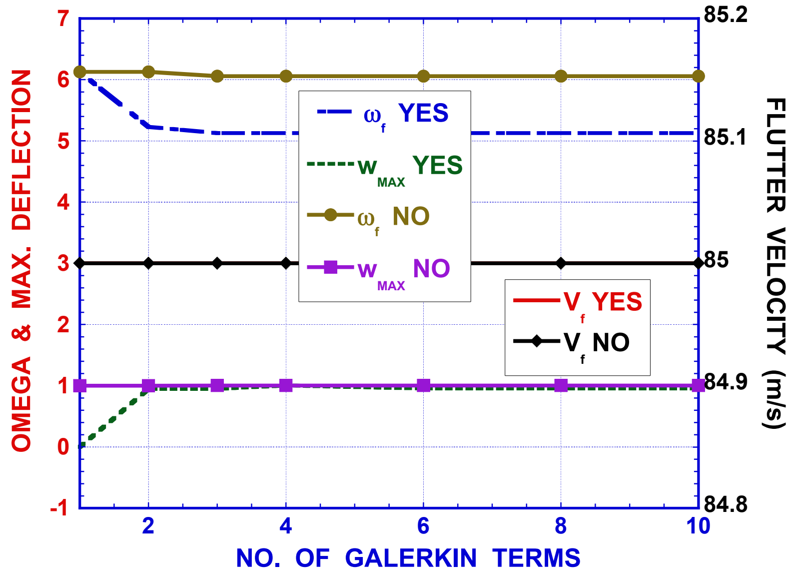

Figure 6: Galerkin series convergence.

Normal to the airflow, the curvature in the x_2-direction is of limited significance since its presence with or without the use of the Galerkin protocol only changes the values of the coefficients in the governing relations. Similar or equal changes in these coefficients can also be produced through normal variations of standard parameters, such as stiffness, inertia, aerodynamics, etc. For the simply supported edges parallel to the x_1-axis only a single half sine wave $(\sin \pi x_2/b)$ is required to satisfy the BCs along those plate edges, as seen in Eq. (27).

As a check on possible divergence occurrences, attempts were made to find values of $V^E > 0$ by setting $\widehat{d} = \omega = 0$. None were found by including or excluding the $T_{3.2}$ and $T_{3.4}$ terms, indicating that the elastic eigenvalues V_f and ω_f represent panel flutter. These results are, of course, predicated on the use of a sufficient number of terms in the truncated deflection series (27) to assure proper convergence.

For the examples considered here, the presence of the spatial derivatives influences the results in the subsonic cases, but in the supersonic flow cases.

3.5 Low and high speed viscoelastic panel flutter

Table 5 depicts combined (converged) results for identical elastic or viscoelastic panels subjected to subsonic or supersonic aerodynamic pressures. These results are, of course, strongly influenced by the choice of parameters and their interaction of the various forces, i.e. their phase relations, with each other. Consequently, any conclusions must be considered specific to only the present parameter set and do not lead to any further possible generalizations. Similar conditions were noted for the viscoelastic instabilities in [32, 33, 10, 81, 61, 44]

The present limited results indicate that the viscoelastic flutter velocities are generally lower than their elastic counters parts, while the flutter frequencies are nearly equal. Thus indicating that in this instance the additional presence of viscoelastic damping had a destabilizing effect. However, different value combinations of the various parameters can produce different results with increased flutter velocities and associated stabilizing effects.

For viscoelastic flutter velocities and frequencies, the Galerkin series converged with four terms as seen in Fig. 6 and in Tables 3 and 4.

Table 5: Panel flutter conditions.

$\tau(s)$	$\frac{E_\infty}{E_0}$	Term $T_{3.2}$ & $T_{3.4}$ Coefficients A_2 and A_4 in Eq. (14) (x_1 derivatives)	Subsonic		Supersonic	
			V_f (m/s) / M_f	ω_f (Hz)	V_f (m/s) / M_f	ω_f (Hz)
0	1	$\neq 0$ (included)	85 / 0.25	5.13	1410 / 4.15	52.96
0	1	$= 0$ (excluded)	85 / 0.25	6.06	1390 / 4.09	51.57
1	.5	$\neq 0$ (included)	80 / 0.24	7.844	485 / 1.41	30.76
10	.5	$\neq 0$ (included)	80 / 0.24	7.843	485 / 1.41	30.76
10^2	.5	$\neq 0$ (included)	80 / 0.24	7.833	485 / 1.41	30.76
10^3	.5	$\neq 0$ (included)	85 / 0.25	5.128	485 / 1.41	30.76
10^4	.5	$\neq 0$ (included)	85 / 0.25	5.128	485 / 1.41	30.76
10^5	.5	$\neq 0$ (included)	85 / 0.25	5.128	485 / 1.41	30.76
1	.1	$\neq 0$ (included)	55 / 0.16	15.37	485 / 1.41	30.76
10	.1	$\neq 0$ (included)	55 / 0.16	15.37	485 / 1.41	30.76
10^2	.1	$\neq 0$ (included)	60 / 0.18	14.27	485 / 1.41	30.76
10^3	.1	$\neq 0$ (included)	80 / 0.24	7.843	485 / 1.41	30.76
10^4	.1	$\neq 0$ (included)	85 / 0.25	5.128	485 / 1.41	30.76
10^5	.1	$\neq 0$ (included)	85 / 0.25	5.128	485 / 1.41	30.76
1	.5	$= 0$ (excluded)	85 / 0.25	6.06	1365 / 3.96	51.75
10	.5	$= 0$ (excluded)	85 / 0.25	6.06	1365 / 3.96	51.75
10^2	.5	$= 0$ (excluded)	85 / 0.25	6.06	1365 / 3.96	51.75
10^3	.5	$= 0$ (excluded)	85 / 0.25	6.06	1365 / 3.96	51.75
10^4	.5	$= 0$ (excluded)	85 / 0.25	6.06	1365 / 3.96	51.75
10^5	.5	$= 0$ (excluded)	85 / 0.25	6.06	1365 / 3.96	51.75
1	.1	$= 0$ (excluded)	70 / 0.20	12.64	1365 / 3.96	51.75
10	.1	$= 0$ (excluded)	85 / 0.25	6.06	1365 / 3.96	51.75
10^2	.1	$= 0$ (excluded)	85 / 0.25	6.06	1365 / 3.96	51.75
10^3	.1	$= 0$ (excluded)	85 / 0.25	6.06	1365 / 3.96	51.75
10^4	.1	$= 0$ (excluded)	85 / 0.25	6.06	1365 / 3.96	51.75
10^5	.1	$= 0$ (excluded)	85 / 0.25	6.06	1365 / 3.96	51.75

4. Conclusions

The analysis of a simple linear isothermal isotropic elastic and viscoelastic flat plate without any external controls or thermal moments reveals that panel flutter in the form of SHM is possible at any velocity from incompressible to supersonic. Even though the elastic-viscoelastic correspondence principle applies to stresses, strains and deflections, the viscoelastic panel flutter velocities bear no relation to their elastic counterparts. They can be $V_f \gtreqless V_f^E$ depending on phase relationships between the inertia, viscoelastic, thermal, in-plane, control and aerodynamic forces. Consequently, more complicated linear or nonlinear configurations and viscoelastic materials will also lead to panel fluter in the same flight regimes.

With converging deflection series, no positive velocity eigenvalues were found at zero frequency, even though the elastic eigenvalue iterative solution protocol was deliberately started by specifying $\omega = 0$. These results indicate that only panel flutter and an absence of panel divergence conditions are achieved, provided a sufficient number of terms are used in the deflection series to assure convergence. Flutter velocities, flutter frequencies and panel deflections each required a different number of terms in the truncated deflection series expression.

The caveat for self excited closed loop systems is that more damping, mass, lift, control and/or stiffness will not necessarily produce larger flutter velocities for any structural material. The proper clues lie in the phase relations between aerodynamic, inertia and viscoelastic forces and require examination on a case by case basis.

In the final analysis, the solution of the governing relations is at the mercy of the chosen model which does not necessarily represent the real world in its entirety.

Additionally, the panel may also experience creep buckling, material failures, aging, etc., where any one may occur time-wise ahead of the others including flutter or quasi-static divergence of panels and lifting surfaces, thus making the earliest occurrence the analysis/design criterion of necessity and choice.

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A Aerodynamic pressure function examples

Several representative examples for the flexible body unsteady aerodynamic pressure $q_P(\widetilde{x}, t)$ of Eq. (14) are provided in [21] For low Mach number M subsonic flow in the x_1-direction it can be approximated by

$$q_P(\widetilde{x}, t) \approx A_{00}(x_1) \underbrace{\rho_\infty V^2}_{=2\,q_\infty} \left[\frac{\partial^2 w}{\partial x_1^2} + \frac{2}{V} \frac{\partial^2 w}{\partial x_1 \partial t} + \frac{1}{V^2} \frac{\partial^2 w}{\partial t^2} \right] \qquad M < 1 \qquad (41)$$

while for supersonic flow at low reduced frequencies, $a\,\omega/V \ll 1$, the expression becomes

$$q_P(\widetilde{x}, t) \approx \frac{\rho_\infty V^2}{\sqrt{M^2 - 1}} \left[\frac{\partial w}{\partial x_1} + \frac{M^2 - 2}{M^2 - 1} \frac{1}{V} \frac{\partial w}{\partial t} \right] \qquad M > 1 \qquad (42)$$

or the so-called "piston theory" approximation can be used to yield

$$q_P(\widetilde{x}, t) \approx \frac{\rho_\infty V^2}{M} \left[\frac{\partial w}{\partial x_1} + \frac{1}{V} \frac{\partial w}{\partial t} \right] \qquad M > 1 \tag{43}$$

The function A_{00} is defined as

$$A_{00}(x_1) = \frac{1}{\pi} \int_{x_1/a}^{1-x_1/a} \ln|y| \; dy \qquad 0 \le x_1 \le a \tag{44}$$

B All inclusive plate governing relations

For small deflections, additional terms can be included in the governing relations to account for non-homogeneous anisotropic behavior, in-plane loads, thermal and in-plane loads, aerodynamic noise and control forces. Such an "all inclusive" formulation produces a considerably more complex plate equilibrium equation[13]

$$\mathcal{L}_{PL}\Big\{ w(x_1, x_2, t) \Big\} = \underbrace{\rho_{PL}(x) \frac{\partial^2 w(x,t)}{\partial t^2}}_{\text{inertia } (T_{1PL})} + \overbrace{\underbrace{c_{VD}(x,t) \frac{\partial w(x,t)}{\partial t}}_{\text{viscous mechanical damping } (T_{PL})} + \underbrace{\imath\, g\, E_0(x)\; w(x,t)}_{\text{structural damping = Coulomb (dry) friction } (T_{3PL})}}^{\text{external contributions}}$$

$$+ \underbrace{\int_{-\infty}^{t} \frac{\partial^2}{\partial x_1^2} \left(D^*_{1111}(x, t-t') \frac{\partial^2 w(x,t')}{\partial x_1^2} \right) dt'}_{\text{elastic or viscoelastic bending resistance } (T_{4APL})}$$

$$+ \underbrace{\int_{-\infty}^{t} \frac{2\, \partial^2}{\partial x_1 \partial x_2} \left(\Big[D^*_{1212}(x, t-t') + 2\, D^*_{2323}(x, t-t') \Big] \frac{\partial^2 w(x,t')}{\partial x_1 \partial x_2} \right) dt'}_{\text{elastic or viscoelastic bending resistance } (T_{4BPL})}$$

$$+ \underbrace{\int_{-\infty}^{t} \frac{\partial^2}{\partial x_2^2} \left(D^*_{2222}(x, t-t') \frac{\partial^2 w(x,t')}{\partial x_2^2} \right) dt'}_{\text{elastic or viscoelastic bending resistance } (T_{4CPL})}$$

$$- \left[\underbrace{\overbrace{\int_{-\infty}^{t} \int_0^a D^*_{1111}(\overline{x}_1, x_2, t-t') \left(\frac{\partial w(\overline{x}_1, x_2, t')}{\partial \overline{x}_1} \right)^2 d\overline{x}_1\, dt'}^{\text{nonlinear contribution}}}_{\text{in plane force due to length change in } x_1 \text{ direction } (T_{5PL})} + \underbrace{\mathcal{N}^{\mathbf{EX}}_{11}(x_2, t)}_{\text{external force } (T_{6PL})} \right] \frac{\partial^2 w}{\partial x_1^2}$$

$$+ \left[\underbrace{\overbrace{\int_{-\infty}^{t} \int_0^b D^*_{2222}(x_1, \overline{x}_2, t-t') \left(\frac{\partial w(x_1, \overline{x}_2, t')}{\partial \overline{x}_2} \right)^2 d\overline{x}_2\, dt'}^{\text{nonlinear contribution}}}_{\text{in plane force due to length change in } x_2 \text{ direction } (T_{7PL})} + \underbrace{\mathcal{N}^{\mathbf{EX}}_{22}(x_1, t)}_{\text{external force } (T_{8PL})} \right] \frac{\partial^2 w}{\partial x_2^2}$$

[13] The differential form may be obtained through formal derivation or by simply replacing the integral expressions with appropriate $\boldsymbol{Q}$ operators and all other terms should have proper $\boldsymbol{P}$ added [34] Also see Eqs. (11) and (16).

$$
- \left[\underbrace{\overbrace{\int_{-\infty}^{t} \int_{0}^{a} D^{*}_{1212}(\overline{x}_1, x_2, t-t') \frac{\partial w(\overline{x}_1, x_2, t')}{\partial \overline{x}_1} \frac{\partial w(\overline{x}_1, x_2, t')}{\partial x_2} \, d\overline{x}_1 \, dt'}^{\text{nonlinear contribution}}}_{\text{in plane force due to angle change between } x_1 \text{ \& } x_2 \text{ directions } (T_{9PL})} + \underbrace{\mathcal{N}^{\mathbf{EX}}_{12}(t)}_{\substack{\text{external force} \\ (T_{10PL})}} \right] \frac{\partial^2 w}{\partial x_1 \partial x_2}
$$

$$
- \left[\underbrace{\overbrace{\int_{-\infty}^{t} \int_{0}^{b} D^{*}_{1212}(x_1, \overline{x}_2, t-t') \frac{\partial w(x_1, \overline{x}_2, t')}{\partial x_1} \frac{\partial w(x_1, \overline{x}_2, t')}{\partial \overline{x}_2} \, d\overline{x}_2 \, dt'}^{\text{nonlinear contribution}}}_{\text{in plane force due to angle change between } x_1 \text{ \& } x_2 \text{ directions } (T_{11PL})} + \underbrace{\mathcal{N}^{\mathbf{EX}}_{12}(t)}_{\substack{\text{external force} \\ (T_{12PL})}} \right] \frac{\partial^2 w}{\partial x_1 \partial x_2}
$$

$$
+ \underbrace{\underbrace{\widehat{a}_0(x_2)}_{= dC_l/d\alpha} \underbrace{\frac{\rho_\infty \, V^2}{2}}_{= q_\infty} \sin \left\{ \frac{\pi}{2 \, \alpha_{ST}(x_2)} \left[\underbrace{\overbrace{\mathcal{A}_W(\alpha, \theta, W)}^{\substack{\text{wing contribution} \\ \text{Eq. (46)}}} + \overbrace{\arctan \left[\mathcal{A}_P(\widetilde{x}, t, w) \right]}^{\substack{\text{panel contribution} \\ \text{Eq. (47)}}}}_{\substack{\text{combined angle of attack due to} \\ \text{deformed wing and panel}}} \right] \right\}}_{\text{lift forces } (T_{13PL})}
$$

$$
+ \underbrace{\mathcal{N}^{\mathbf{P}}_{11}(x,t) \frac{\partial^2 w}{\partial x_1^2}}_{x_1 \text{ piezo force } (T_{14PL})} + \underbrace{\mathcal{N}^{\mathbf{P}}_{22}(x,t) \frac{\partial^2 w}{\partial x_2^2}}_{x_2 \text{ piezo force } (T_{15PL})} + \underbrace{\mathcal{N}^{\mathbf{T}}_{11}(x,t) \frac{\partial^2 w}{\partial x_1^2}}_{x_1 \text{ thermal force } (T_{16PL})} + \underbrace{\mathcal{N}^{\mathbf{T}}_{22}(x,t) \frac{\partial^2 w}{\partial x_2^2}}_{x_2 \text{ thermal force } (T_{17PL})}
$$

$$
+ \underbrace{2 \, \mathcal{N}^{\mathbf{T}}_{12}(x,t) \frac{\partial^2 w}{\partial x_1 \partial x_2}}_{\substack{\text{thermal shear} \\ \text{force } (T_{18PL})}} + \underbrace{\frac{\partial M^{\mathbf{T}}_{11}}{\partial x_2^2}}_{\substack{x_3 \text{ load due} \\ \text{to } M^{\mathbf{T}}_{11} (T_{19PL})}} + \underbrace{\frac{\partial M^{\mathbf{T}}_{22}}{\partial x_1^2}}_{\substack{x_3 \text{ load due} \\ \text{to } M^{\mathbf{T}}_{22} (T_{20PL})}} + \underbrace{2 \, \frac{\partial M^{\mathbf{T}}_{12}}{\partial x_1 \partial x_2}}_{\substack{x_3 \text{ load due} \\ \text{to } M^{\mathbf{T}}_{12} (T_{21PL})}}
$$

$$
+ \underbrace{\Delta p \left(q, x, t, \alpha, \overbrace{w, \frac{\partial w}{\partial x_1}, \frac{\partial^2 w}{\partial x_1 \, \partial t}, \frac{\partial w}{\partial t}, \frac{\partial^2 w}{\partial t^2}}^{\text{panel contributions}}, \overbrace{\theta, \frac{\partial \theta}{\partial t}, \frac{\partial^2 \theta}{\partial t^2}, W, \frac{\partial W}{\partial t}, \frac{\partial^2 W}{\partial t^2}}^{\text{wing contributions}} \right)}_{\text{aerodynamic noise pressure } (T_{22PL})} + \underbrace{F_V(x,t)}_{\substack{\text{vibratory} \\ \text{force } (T_{23PL})}}
$$

$$
+ \underbrace{F_{SC} \left(\overbrace{x, t, w(x,t), \frac{\partial w(x,t)}{\partial t}, \frac{\partial^2 w(x,t)}{\partial t^2}, \frac{\partial^3 w(x,t)}{\partial t^3}, \int_0^t w(x,t') \, dt'}^{\text{proportional } (T_{24P}), \text{ integral } (T_{24I}) \text{ and/or differential } (T_{24D}) \text{ controller}} \right)}_{\text{external closed loop servo}-\text{control force } (T_{24PL})}
$$

$$
+ \underbrace{F_C \left(\overbrace{\boldsymbol{V}(x,t,w)}^{\substack{\text{piezo}- \\ \text{electric} \\ \text{voltage } (T_{25PZ})}}, \overbrace{\boldsymbol{I}(x,t,w)}^{\substack{\text{MR} \\ \text{current} \\ (T_{25MR})}}, \overbrace{\boldsymbol{\sigma}^{\text{SM}}(x,t,w)}^{\substack{\text{smart} \\ \text{materials} \\ (T_{25SM})}} \right)}_{\text{external open or closed loop control force } (T_{25PL})} + \underbrace{F_{IP}(x,t)}_{\substack{\text{cabin} \\ \text{presurization } (T_{26PL})}} = 0 \quad (45)
$$

with $\imath = \sqrt{-1}$ and where deflections for the plate $w = w(\widetilde{x}, t) = w(x_1, x_2, t)$ and for the wing[14] are $W = W(x_2, t)$ unless otherwise indicated. The angle $\alpha_{ST}(x_2)$ is the section stall angle. Terms T_{5PL}, T_{7PL}, T_{11PL} and T_{16PL} through T_{21PL} are inherently nonlinear.

In the absence of chordwise bending the panel effective angle of attack due to wing contributions is

$$\mathcal{A}_W(\alpha,\theta,W) = \overbrace{\alpha_r(x_2)-\alpha_0(x_2)}^{\substack{f_\alpha(x_2)=\text{built in}\\ \text{rigid angles}}} + \overbrace{\alpha(x_2)}^{\substack{\text{angle of}\\ \text{attack}}} + \underbrace{\overbrace{\theta(x_2,t)}^{\substack{\text{wing angle}\\ \text{of twist}}} + \overbrace{\arctan\left(\frac{1}{U_\infty}\frac{\partial W(x_2,t)}{\partial t}\right)}^{\text{wing bending contribution}}}_{\text{wing aero}-\text{viscoelastic contributions}} \tag{46}$$

and the angle of attack due to panel deflections is a selective combination of the following terms (See Eqs. (41) to (43).)

$$\mathcal{A}_P(\widetilde{x},t,w) = \arctan\left\{\frac{\partial w}{\partial x_1}+\frac{1}{V}\frac{\partial w}{\partial t}+a\left[\frac{\partial^2 w}{\partial x_1^2}+\frac{1}{V^2}\frac{\partial^2 w}{\partial t^2}+\frac{1}{V}\frac{\partial^2 w}{\partial x_1 \partial t}\right]\right\} \tag{47}$$

The bending rigidities D_{ijkl} are defined as

$$D^*_{ijkl}(\widetilde{x},t-t') = D^*_{ijkl}(x_1,x_2,t-t') = \int_{-h/2}^{h/2} E^*_{ijkl}(x,t-t')\; x_3^2\; dx_3 \tag{48}$$

Terms such as T_{4APL} of Eqs. (45) and all others which include D^*_{ijkl} terms can be integrated by parts to yield

$$\underbrace{\int_{-\infty}^{t}\frac{\partial^2}{\partial x_1^2}\left(D^*_{1111}(x,t-t')\,\frac{\partial^2 w(x,t')}{\partial x_1^2}\right)dt'}_{\text{elastic or viscoelastic bending resistance }(T_{4APL})} =$$

$$\underbrace{\frac{\partial^2}{\partial x_1^2}\left(D_{1111}(x,0)\,\frac{\partial^2 w(x,t)}{\partial x_1^2}\right)}_{\text{instantaneous elastic response }(T_{4AELPL})} + \underbrace{\int_{0}^{t}\frac{\partial^2}{\partial x_1^2}\left(D_{1111}(x,t-t')\,\frac{\partial^3 w(x,t')}{\partial x_1^2\partial t'}\right)dt'}_{\text{viscoelastic creep/relaxation response }(T_{4AVEPL})} \tag{49}$$

When unsteady thermal conditions due to temperatures $T(x,t)$ are included, then all $D^*_{ijkl}(\widetilde{x},\, t-t')$ must be replaced by $D^*_{ijkl}(\widetilde{x},t,t')$ in all above relations, with similar substitutions for the $E^*_{ijkl}(\widetilde{x},t-t')$. This eliminates the presence of convolution time integrals in the constitutive relations (1) and wherever they have been applied and changes these relations to

$$\sigma_{kl}(x,t) = \underbrace{\int_{-\infty}^{t} E^*_{klmn}\left[x,t,t',T(x,t')\right]\;\epsilon_{mn}(x,t')\;dt'}_{\text{stresses generated by ordinary strains}} - \underbrace{\int_{-\infty}^{t} E^{T*}_{kl}\left[x,t,t',T(x,t')\right]\;\alpha T(x,t')\;dt'}_{\text{thermal stresses}} \tag{50}$$

which for static temperatures reduce to

$$\sigma_{kl}(x,t) = \int_{-\infty}^{t} E^*_{klmn}\left[x,t-t',T(x)\right]\;\epsilon_{mn}(x,t')\,dt' - \int_{-\infty}^{t} E^{T*}_{kl}\left[x,t-t',T(x)\right]\;\alpha T(x)\,dt' \tag{51}$$

[14]generic designation for any lifting surface, such as wing, tail, fuselage, flap, aileron, etc.

where α is the coefficient of thermal expansion with dimensions $[L/(^{\circ}K\ L)]$, with $^{\circ}K$ denoting degrees Kelvin or Centigrade.

The inclusion of large deformations and/or curved plates fundamentally alters Terms T_{4PL} in Eq. (45) and requires the use of an additional governing relation [16]

C Modifications to MATLAB™ eigenvector subroutine POLYEIG

The Subroutine POLYEIG [68] as provided by MATLAB™ was written to return eigenvectors which have the lowest errors. Unfortunately those are not necessarily the desired ones corresponding to their lowest positive values.[15] In the currently considered example the subroutine consistently returned zero values for all the coefficients B_{mn} of the deflection series (26). This phenomenon manifest itself for the cases when the x_1 derivatives are absent in (14) but it did not occur when the derivatives are included. One reason why in the latter case the zero displacement solution is not necessarily the one with the lowest errors is because the selected Galerkin sine series does not satisfy the governing relation term by term when the cosine terms stemming from the x_1 derivatives are included.

Consequently, the native MATLAB™ subroutine was modified to return the lowest positive eigenvectors, i.e. the principal eigenvalues. On the other hand, the V_f and ω_f eigenvalue pairs were computed by a separate non-library subroutine especially developed to return the lowest positive velocity eigenvalues with their paired frequencies.

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